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PLANE BOUNDARY-VALUE PROBLEMS FOR THE SINE-HELMHOLTZ EQUATION IN THE THEORY OF ELASTICITY OF LIQUID CRYSTALS IN NON-UNIFORM MAGNETIC FIELDS†

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It is shown that, with certain limitations, the equations of orientational equilibrium of a nematic liquid crystal in the two-dimensional domain, placed in a non-uniform magnetic field, reduce to the non-linear sine-Helmholtz equation in the plane of conjugate magnetic potentials μ and η. The part played by the conformal mappings $\mu, \eta \to x, y$ is investigated. Plane boundary-value problems for this equation in the one- and two-dimensional regions are considered. Criteria of stability of the two-dimensional solutions in open and closed volumes are established. The explicit forms of the solutions, which are expressed in terms of periodic elliptic functions and quasi-periodic theta functions of two arguments, are analysed. The inverse problem is solved. Solutions of the two-dimensional kink and the Jacobi delta-function type in a closed volume are obtained.

1. Nematic liquid crystals, possessing liquid mobility, exhibit an unusual orientational elasticity. In pure form it is found in fixed nematic liquid crystals subjected to a moment due to a magnetic or electric field and solid boundaries rather than to a force. These factors produce a spatially non-uniform director field $\mathbf{l}(x, y, z)$ in the crystal. The unit vector $\mathbf{l}$ represents the mean statistical direction (in a physically small volume) of the axes of elongated (oblate) molecules, between which there are long-range moment interactions.

The equation of equilibrium of the local moments in a fixed nematic liquid crystal can be obtained from the variational principle. The Oseen–Frank elastic energy

$$F=\frac{1}{2}\int_{\upsilon}[k_1(\operatorname{div}\mathbf{l})^2+k_2(\mathbf{l}\operatorname{rot}\mathbf{l})^2+k_3|\mathbf{l}\times\operatorname{rot}\mathbf{l}|^2+\chi_{ik}H_iH_k]d\upsilon \tag{1.1}$$

is related to the gradients of the field $\mathbf{l}(x,y,z)$. This does not include the energy of the change in volume υ. The coefficients k_1 and k_3 are the elastic moduli of the bending of the lines of force of the orientational field, k_2 is the torsional modulus and H_i is the magnetic field vector. The diamagnetic tensor $\chi_{ik} = \chi_\perp\delta_{ik} + (\chi_\parallel - \chi_\perp)l_il_k$ has two components—a longitudinal component (χ_{11}) and a transverse component (χ_1) of the susceptibility of the nematic liquid crystal.

We will consider the magnetic field as being quasi-periodic, obeying the following equations

$$\operatorname{div}\mathbf{B} = 0,\ \operatorname{rot}\mathbf{H} = 0,\quad \mathbf{H} = \operatorname{grad}\mu,\ \mathbf{B} = \mathbf{H} \tag{1.2}$$

Although we must take $B_i = \chi_{ik}H_k$ for the magnetic induction vector, in view of the diamagnetism of the nematic liquid crystal the magnetic polarizabiity of the medium is extremely small. Hence, the last relation in (1.2), which is in fact an assumption of the theory, is satisfied in reality with high accuracy, of the order of 10^{-7}.

We must make other assumptions in addition to this one. We will assume, first, that

$$k_1 \simeq k_3 = k \tag{1.3}$$

although nematic liquid crystals exist for which this relation holds accurately. Second, we will confine ourselves to plane bending problems, ignoring twisting of the lines of force, by putting $\mathbf{l}\operatorname{rot}\mathbf{l} = 0$. We will introduce the angle functions $\varphi(x, y)$ and $\psi(x, y)$ which represent the orientation of the vectors $\mathbf{l}$

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and **H** in the (x, y) plane with respect to the OY axis (a Cartesian system of coordinates with origin at an arbitrary point O), given by the following expressions

$$l_x = \sin\varphi, \quad l_y = \cos\varphi, \quad H_x = H\sin\psi, \quad H_y = H\cos\psi \tag{1.4}$$

With these assumptions, the following equation of equilibrium of the moments in a nematic liquid crystal corresponds to functional (1.1)

$$k\nabla^2(2\alpha) - H^2\chi\sin 2\alpha = -k\nabla^2 2\psi, \quad \alpha = \varphi - \psi, \quad \chi = \chi_\parallel - \chi_\perp \tag{1.5}$$

The angle of orientation of the magnetic field is

$$\psi = \operatorname{arctg}(\mu'_x / \mu'_y), \quad \mu'_x = \partial\mu / \partial x, \quad \mu'_y = \partial\mu / \partial y, \quad \nabla^2\mu = 0 \tag{1.6}$$

i.e. it is found from the equations of the magnetic field, which reduce to Laplace's equation for the magnetic potential μ.

The angles of orientation are specified at the boundary of the two-dimensional region S as a function of the coordinates of the boundary

$$\alpha = \alpha_s, \quad \psi = \psi_s \tag{1.7}$$

The boundary-value problem (1.5)–(1.7) enables us to obtain the orientational field $\mathbf{l}(x, y)$ in the region S, which is generated under the influence of the magnetic field $\mathbf{H}(x, y)$ and the specified orientation at the boundary. Despite these assumptions, the problem turns out to be quite complex in view of the singularities of the non-linear equation (1.5). It has so far only been investigated in the one-dimensional case (see the review in [3]).

Below, we develop a general approach to solving two-dimensional problems.

2. We reduce (1.5) to a form which does not contain inhomogeneous terms. By a conformal transformation of the x, y coordinates to orthogonal curvilinear coordinates $\mu(x, y)$, $\eta(x, y)$, which are real and imaginary conjugate magnetic potentials, the two-dimensional Laplace operator ∇^2 can be converted as follows:

$$\nabla^2 \to H^2\nabla^2_\mu, \quad \nabla^2_\mu \equiv \partial^2 / \partial\mu^2 + \partial^2 / \partial\eta^2, \quad \nabla^2 \equiv \partial^2 / \partial x^2 + \partial^2 / \partial y^2 \tag{2.1}$$

The use of this formula in (1.5) clearly enables one to dispense with the variable factor H^2 in the equation. Further, since the potentials μ and η are conjugate harmonic functions satisfying Laplace's equation and the Cauchy–Riemann relations

$$\nabla^2\mu = \nabla^2\eta = 0, \quad \mu'_x = \eta'_y, \quad \eta'_x = -\mu'_y \tag{2.2}$$

it can be proved [4], that the angle ψ of orientation of the vector of the quasi-potential magnetic field is also a harmonic function satisfying Laplace's equation

$$\nabla^2\psi = 0 \tag{2.3}$$

This enables us to equate the right-hand side in (1.5) to zero.

In view of the importance of relation (2.3), which is unknown in field theory, we will spend some time on proving it. Differentiating the formula for the magnetic field in (1.4) and using the magnetic-field equations (1.2), we obtain

$$\psi'_x = -\frac{1}{2}(\ln H^2)'_y, \quad \psi'_y = \frac{1}{2}(\ln H^2)'_x \tag{2.4}$$

(the primes denote partial derivatives with respect to the variables indicated). Further, differentiating the first relation with respect to y and the second relation with respect to x and adding the equations obtained, we arrive at (2.3). This result was previously obtained in [5] in a more complex way.

Thus, using (2.1) and (2.3) we can write the orientational equation (1.5) in the following final form [5]

$$\mu_\chi^2\nabla_\mu^2(2\alpha)-\sin 2\alpha=0,\quad \mu_\chi=\sqrt{k/\chi} \tag{2.5}$$

(μ_χ is the characteristic magnetic potential). Another form of writing (2.5), expressed in terms of the additional angle β, is

$$\mu_\chi^2\nabla_\mu^2(2\beta)+\sin 2\beta=0,\quad 2\beta=\pi-2\alpha \tag{2.6}$$

The new angular function $\beta(x, y)$ is more convenient in certain boundary-value problems, since it varies in the same way as the orientational deformations $\nabla\mathbf{l}$. It is important that, as a result of conformal transformations, the angles α and β are preserved, while (1.5) is converted to a simpler non-linear equation (homogeneous with constant coefficients). It enables us to obtain initially the angular function $\alpha(\mu, \eta)$ or $\beta(\mu, \eta)$ in the space of the variables μ and η. Then, after solving the corresponding magnetic problem we must transform α and β into the real space x, y, knowing the magnetic potentials $\mu(x, y)$ and $\eta(x, y)$. The initial angle functions φ and ψ, which define the orientation of the vectors **l** and **H**, are then found from the formulae

$$\varphi(x, y)=\alpha[\mu(x, y), \eta(x, y)]+\psi(x, y) \tag{2.7}$$

$$\psi(x,y)=\operatorname{arctg}(\mu_x'/\mu_y'),\quad \mu=\mu(x,y),\quad \eta=\eta(x,y) \tag{2.8}$$

The general solutions (2.7) and (2.8) specify a relatively simple local relationship between the fields $\mathbf{l}(x,y)$ and $\mathbf{H}(x,y)$. The functions $\alpha(\mu, \eta)$ and $\beta(\mu, \eta)$ are universal for a whole class of magnetic problems, the solutions of which $\mu(x,y)$, $\eta(x,y)$ conformally transform the actual boundaries of the region (in the (x, y) plane) into the same boundary in the (μ, η) plane. The specific generating function $\alpha(\mu, \eta)$ or $\beta(\mu, \eta)$ is constructed once for this boundary.

The subsequent material of this paper is devoted to the construction of these functions.

We note one important consequence of (2.7) and (2.8). In addition to the direct problem of finding the orientational fields $\mathbf{l}(x,y)$ from the external magnetic field $\mathbf{H}(x,y)$ and the boundary conditions, we can also solve the inverse problem, namely, to find the magnetic field from the orientational field that exists in equilibrium with it.

Differentiating $\alpha[\mu(x,y), \eta(x,y)]$ as a complex function and using relations (1.2) and (2.2), we obtain initially

$$H_x=\frac{\alpha_x''/\alpha_\eta'-\alpha_y'/\alpha_\mu'}{\alpha_\mu'/\alpha_\eta'+\alpha_\eta'/\alpha_\mu'},\quad H_y=\frac{\alpha_x'/\alpha_\mu'+\alpha_y'/\alpha_\eta'}{\alpha_\mu'/\alpha_\eta'+\alpha_\eta'/\alpha_\mu'} \tag{2.9}$$

The derivatives α_μ' and α_η' can in many cases be expressed as first integrals in terms of quadratic functions of the angle α. In (2.9) we can change from the angles α and their gradients to the absolute angles φ and ψ, if we use (2.7), (2.8) and (2.4). Naturally, the inverse formulae also have a local character. They constitute the theoretical basis for a method of monitoring the magnetic fields using the orientational field of the nematic liquid crystal, which in turn can be monitored by polarization-optical methods [3, 6].

3. The boundary-value problems for the non-linear equation (2.5) or (2.6) have apparently not been previously considered. Only recently have fairly general solutions of it begun to appear in the literature. We will consider them as they apply to two-dimensional (plane) problems of the theory of elasticity of nematic liquid crystals. We will confine ourselves here to the condition of uniform orientation of the nematic liquid crystal along all the boundaries or its individual parts. This is most often of all realized in practice. With this assumption we will consider two classes of problems.

The first of these is related to the regions whose boundaries are either equipotentials (μ = const) of the magnetic field, or coincide with the lines of force (η = const). Then, Eq. (2.5) or (2.6) must be solved in the plane of the variables (μ, η) for the strip $\mu_1 \leqslant \mu \leqslant \mu_2$ or $\eta_1 \leqslant \eta \leqslant \eta_2$ with the boundary conditions

$$\beta(\mu_1)=\beta_1,\quad \beta(\mu_2)=\beta_2 \quad \text{or} \quad \beta(\eta_1)=\beta_1,\quad \beta(\eta_2)=\beta_2 \tag{3.1}$$

where β_1 and β_2 are constants which define the orientation of the unit vector **l** at the boundary with respect to the normal or with respect to the boundary itself. In a similar way we can also investigate the conditions for the supplementary angle α. The one-dimensional solution of the form $\beta(\mu)$ or $\beta(\eta)$ obviously satisfies boundary conditions (3.1). This solution turns out to be two-dimensional in the actual space (x, y) if the magnetic problem is two-dimensional: $\mu = \mu(x, y)$ or $\eta = \eta(x, y)$.

Another class of more complex but also solvable problems arises with regions some parts of the boundaries of which coincide with equipotentials, while the others coincide with the lines of force of the magnetic field. In this case the two-dimensional equation (2.5) or (2.6) must be solved for a closed region that is rectangular in the (μ, η) plane, the sides of which are parallel to the $O\eta$ or $O\mu$ coordinate axes. Even in the case of constant values of the angle on each of the rectilinear parts, when

$$\beta(\mu_1) = \beta_1, \quad \beta(\mu_2) = \beta_2, \quad \beta(\eta_1) = \beta_3, \quad \beta(\eta_2) = \beta_4 \tag{3.2}$$

we must consider the solution which depends on both potentials μ and η. However, here we will use a distinctive method of separation of variables.

We will consider both these classes of problems in succession.

4. In the first class of problems the one-dimensional equation which follows from (2.5) or (2.6) is solved at the first stage. Bearing this in mind, we write

$$\mu_\chi^2 \partial^2(2\beta)/\partial\mu^2 + \sin 2\beta = 0 \tag{4.1}$$

We will first consider the case when, if there is no magnetic field, there are also no distortions of the orientational field, i.e. the angles on both boundaries are simply the same $\beta_1 = \beta_2$. The following boundary conditions are typical

$$\beta_1(\mu_1) = \beta_2(\mu_2) = 0, \quad \mu_1 = 0, \quad \mu_2 = \Delta\mu \tag{4.2}$$

When there is no magnetic field the angle β is also equal to zero over the whole band. This can be seen from Eq. (4.1).

When there is a magnetic field present, Eq. (4.1) has a family of solutions

$$\sin\beta = \pm \nu\, \mathrm{sn}[\mu_\chi^{-1}\mu(x, y)], \quad \nu = \sin\beta_m \tag{4.3}$$

Here β_m is the greatest angle (at the centre of the strip), the two signs correspond to a twist in different directions, and the symbol sn denotes the Jacobi elliptic sine [7]. It vanishes when its argument is equal to the quantity

$$2nK(\nu), \quad K(\nu) \equiv \int_0^{\pi/2} (1 - \nu^2 \sin\beta)^{-1/2} d\beta, \quad n = 0, 1, 2, \ldots \tag{4.4}$$

($K(\nu)$ is the complete elliptic integral of the first kind and ν is its modulus). Solution (4.3) then satisfies the boundary conditions $\beta_1 = 0$ when $\mu_1 = 0$, when $n = 0$. On the other boundary ($\mu = \mu_2$) condition (4.2) is satisfied if

$$\Delta\mu/\mu_\chi = 2nK(\nu) \qquad n = \pm 1, \pm 2, \ldots \tag{4.5}$$

Hence the constant ν is determined, and, in terms of it, also the greatest angle β_m of twist in the centre of the layer.

By definition [7] $K(\nu)$ is a monotonically increasing function, i.e.

$$K(\nu) > \pi/2, \quad K(0) = \pi/2, \quad K(1) = \infty, \quad 0 \leqslant \nu \leqslant 1 \tag{4.6}$$

Therefore, a non-trivial solution ($\beta = 0$) exists if $\nu = 0$, i.e. $K(\nu) > \pi/2$. From (4.5) and (4.6) we then obtain the inequality

$$|\Delta\mu| > n\pi\mu_\chi, \quad n = 1, 2, \ldots \tag{4.7}$$

Its right-hand side is the threshold value of the potential difference across the boundaries $\Delta\mu$. Below this threshold, we only have the trivial solution $\beta = 0$, i.e. the magnetic field does not distort the field of the director at all and $\mathbf{l} \parallel \mathbf{H}$. When the threshold is reached (the bifurcation point) the undistorted state becomes unstable and the set $(2m)$ of solutions $\beta \neq 0$ becomes possible. The number $|n|$ corresponds to the number of half-waves of the sinusoid which are contained in the interval $\mu_2 - \mu_1 = \Delta\mu$. The two equal and deepest minima of the functional of the elastic energy correspond to the first number $n = \pm 1$, taking the magnetic terms and the two branches of the equilibrium deformations of the director field into account. The branches of higher order $|n| \geqslant 2$ correspond to metastable equilibria [8].

Condition (4.7) is essentially a universal criterion of the occurrence of threshold deformations of the two-dimensional field of the director in the regions of arbitrary geometry with boundaries which are either equipotential lines or magnetic lines of force of the non-uniform magnetic field. This criterion is a generalization of Frederick's criterion [6], which only holds for a plane-parallel layer with homogeneous conditions along it: $\mu = \mu(y)$, $\eta = \text{const}$. The universal nature of criterion (4.7) manifests itself in the fact that only the potential difference and not the linear dimensions of the region occur in it.

Expanding the relation $\mu(x, y)$ in (4.3) as it applies to specific magnetic fields and changing to the angle $\alpha(x, y)$ and Eqs (2.7) and (2.8), we obtain a solution of the problem of finding the fields $\mathbf{l}(x, y)$ and $\mathbf{H}(x, y)$ in final form for conditions (4.2) on the boundaries, excluding the deformation without the magnetic field. Solution (4.3) is also obviously universal with respect to the magnetic fields and the geometry of the region with boundaries $\mu(x, y) = \text{const}$ or $\eta(x, y) = \text{const}$.

We will now consider the case of boundary conditions which, when there is no magnetic field, cause distortions of the director field. In particular, they have the form

$$\beta(\mu_1) = 0, \quad \beta(\mu_0) = \pi/2 \tag{4.8}$$

It is important in principle that the values of the angles at different boundaries of the band should not be the same. "Threshold-free" solutions of Eq. (4.1), in which the distortions of the orientational field are already described for small magnetic fields, satisfy these boundary conditions. They have the form

$$\sin\beta = \pm \operatorname{sn}\left[\frac{\mu(x, y) - \mu_0}{\mu_\chi \nu} \pm K(\nu)\right], \quad 0 \leqslant \nu \leqslant 1 \tag{4.9}$$

The different signs again correspond to twists to the right or to the left. By definition $\operatorname{sn}(\pm K) = \pm 1$, and hence the second boundary condition in (4.6) is obviously satisfied. The zeroth condition in (4.8) is satisfied, in particular, when the argument of the elliptic sine in (4.9) takes the value (4.4). If $n = 0$, i.e. the argument vanishes, we have

$$|\Delta\mu|\mu_\chi^{-1} = \nu K(\nu), \quad \Delta\mu = \mu_1 - \mu_0 \tag{4.10}$$

Taking this into account in (4.9) we obtain the solution in the final form

$$\sin\beta = \pm \operatorname{sn}\left[\frac{\mu_1 - \mu(x, y)}{\Delta\mu} K(\nu)\right] \tag{4.11}$$

Values $|n| = 1$ and above are ignored. They give corrections to the angular field $\beta(\mu)$ up to 180°, which are eliminated in the asymptotic form as $H \to 0$. We will show this.

Relation (4.10) has no form of threshold criterion, since the right-hand side may vanish due to the factor ν.

If $\Delta\mu = 0$ (there is no magnetic field), it follows from (4.10) that $\nu = 0$, since $K(0) = \pi/2$. But then $\operatorname{sn} \to \sin$. Identifying the arguments of the functions on the left and right in (4.11) we then obtain

$$\Delta\mu \to 0, \quad \beta \to \frac{\mu_1 - \mu(x, y)}{\Delta\mu} \frac{\pi}{2} \tag{4.12}$$

This linear relationship between the twist angle and the potential for small values of the magnetic field can also be obtained directly from Eq. (4.1), neglecting the non-linear (magnetic) term in it. The asymptotic expression (4.12) satisfies the boundary conditions (4.8), but eliminates the contributions to 180°, which were correctly neglected above.

The universal formula (4.11), after substituting into it the solutions of the magnetic problems $\mu(x, y)$, obtained for regions bounded by equipotentials gives, after using (2.7) and (2.8), a subclass of solutions of the plane problem of the theory of elasticity of nematic liquid crystals, which describe the effect on the initially non-uniform orientational field $\mathbf{l}(x, y)$ of a non-uniform magnetic field.

Similar results are also obtained for regions bounded by the lines of force of the magnetic field $\mathbf{l}(x, y) = \text{const}$.

In problems of the first class, as in the general case, we can establish an inverse relationship, which expresses the magnetic field in terms of the angle β. Obviously

$$\operatorname{grad}\beta[\mu(x,y)] = \beta'_\mu \operatorname{grad}\mu = \beta'_\mu H \tag{4.13}$$

The derivative β'_μ is the first integral of Eq. (4.1), which has the form

$$\mu_\chi\beta'_\mu = \pm\sqrt{\sin^2\beta_m - \sin^2\beta}, \quad \beta_m = \max\beta \tag{4.14}$$

Equation (4.13) then gives

$$H(x,y) = \pm\mu_\chi(\sin^2\beta_m - \sin^2\beta)^{-1/2}\operatorname{grad}\beta \tag{4.15}$$

This relationship, which is simpler than (2.9), reflects the one-dimensional form of the function $\beta(\mu)$. The gradient of the angle β and the vector H are obviously coaxial. The formula holds for all regions whose boundaries are defined by the equation $\mu(x,y) = \text{const}$ and for arbitrary piecewise-homogeneous boundary conditions.

5. We will now consider the more complex boundary-value problems of the second class for closed regions, bounded simultaneously by equipotentials and lines of force. We will first consider doubly-periodic and then quasi-periodic solutions, which depend on both potentials μ and η.

Doubly-periodic solutions are obtained by the method of separation of variables in Eq. (2.5) or (2.6). In this case we can consider rectangular regions in the plane of these variables, but boundaries in the actual space, part of which coincides with the equipotentials of the magnetic field, and part with the lines of force. In the actual space, regions with curvilinear boundaries also correspond to them. An example is a wedge-shaped cell containing a liquid crystal in which two surfaces (cylindrical) coincide with the lines of force of a circular magnetic field, while two others (radial) are its equipotentials. Similar cells with boundaries of arbitrary geometry can also be conformally mapped onto a rectangular region of the (μ, η) plane, to which the results of this section, given below, will also belong.

The method of separation of variables was developed in [9], though, it is true, for the sine-Gordon equation of similar structures. It is easily modified for the equation considered here.

We will write the solution of Eq. (2.5) initially, as in [9], in the form

$$\operatorname{tg}(\alpha/2) = \alpha_\eta/\alpha_\mu, \quad \alpha_\mu \equiv \alpha(\mu), \quad \alpha_\eta \equiv \alpha(\eta) \tag{5.1}$$

Using the representation $\sin 2\alpha$ in terms of $\operatorname{tg}(\alpha/2)$ and calculating $\nabla^2_\mu(2\alpha)$, we obtain the following equation instead of (2.5)

$$(\alpha_\mu^2 + \alpha_\eta^2)(\alpha''_\mu\alpha_\mu^{-1} - \alpha''_\eta\alpha_\eta^{-1}) + 2(\alpha'_\eta)^2 - 2(\alpha'_\mu)^2 = \alpha_\eta^2 - \alpha_\mu^2 \tag{5.2}$$

Differentiating this equation with respect to μ and with respect to η separately, we obtain two equations with separated variables

$$\frac{1}{\alpha_\mu\alpha'_\mu}\left(\frac{\alpha''_\mu}{\alpha_\mu}\right)' = \frac{1}{\alpha_\eta\alpha'_\eta}\left(\frac{\alpha''_\eta}{\alpha_\eta}\right) = 4a^2$$

where a is an arbitrary constant. Integrating each of these equations twice, we obtain two first-order equations

$$(\alpha'_\eta)^2 = a^2\alpha_\eta^4 + b^2\alpha_\eta^2 + c^2$$

$$(\alpha'_\mu)^2 = a^2\alpha_\mu^4 + (1-b^2)\alpha_\mu^2 + c^2 \tag{5.3}$$

The constants of integration agree with one another by substituting (5.3) into (5.2), and hence, of the six only three (a^2, b^2, c^2) are independent.

In Eq. (2.6) the variables are separated in exactly the same way with α replaced by β in (5.1), but this leads simply to a change in sign in the second terms on the right-hand sides of (5.3).

Integration of Eqs (5.3) with respect to α or of the above equations with respect to β gives a solution in elliptic functions.

We can put $\mathrm{tg}(\alpha/2) = \alpha_\eta \alpha_\mu$ in (5.1). We then obtain equations of the form (5.3), but with α_μ replaced by α_μ^{-1} in the second of them. Each of the forms of the solution has its advantages with regard to some of the boundary conditions, which will be clear from what follows.

The simplest solution in separated variables in the form of a product has the form

$$\mathrm{tg}\frac{\beta}{2} = \pm\sqrt{\nu_\eta \nu_\mu}\,\mathrm{sn}\left[\frac{2\mu(x,y)}{\Delta\mu}K(\nu_\mu)\right]\mathrm{sn}\left[\frac{2\eta(x,y)}{\Delta\eta}K(\nu_\eta)\right] \tag{5.4}$$

The numbers ν_η and ν_μ satisfy the characteristic equations

$$\frac{(1+\nu_\mu^2)}{\Delta\mu^2}K^2(\nu_\mu) + \frac{(1+\nu_\eta^2)}{\Delta\eta^2}K^2(\nu_\eta) = \frac{1}{4\mu_\chi^2} \tag{5.5}$$

$$\frac{\nu_\mu}{\Delta\mu^2}K^2(\nu_\mu) = \frac{\nu_\eta}{\Delta\eta^2}K^2(\nu_\eta)$$

Solution (5.4) satisfies the zeroth boundary conditions with respect to β on the sides of the rectangle $\mu_1, \mu_2, \eta_1, \eta_2$, i.e. the condition $\mathbf{l} \perp \mathbf{H}$ is satisfied on the boundaries. When there is no magnetic field the orientations are also uniform over the whole region.

Taking the properties of the function $K(\nu)$, written in (4.6), into account in (5.5) we arrive at the conclusion that the orientational deformation has a threshold form. It occurs abruptly, when the potential differences $\Delta\mu$ and $\Delta\eta$ satisfy the condition

$$\Delta\mu^{-2} + \Delta\eta^{-2} \leqslant (\pi\mu_\chi)^{-2}, \quad \Delta\eta = |\eta_2 - \eta_1|, \quad \Delta\mu = |\mu_2 - \mu_1| \tag{5.6}$$

The equality sign corresponds to the critical values $\Delta\mu_t$, $\Delta\eta_t$. The bifurcation boundary is now obviously not a point but a circle of radius $(\pi\mu_\chi)^{-1}$ in the plane of the variables $\Delta\mu^{-1}$, $\Delta\eta^{-1}$.

The threshold criterion for two-dimensional deformations has been obtained in this paper for the first time. The non-uniformity of the magnetic field and the closed nature of the region S have been completely taken into account.

The universal nature of the criterion should be noted. It relates to all the curvilinear regions, conformally mapped onto a rectangle in the (μ, η) plane. The boundaries of the regions are specified by the equations $\mu(x, y) = \text{const}$, $\eta(x, y) = \text{const}$, where $\mu(x, y)$ and $\eta(x, y)$ are the conjugate potentials of the non-uniform magnetic field. By specifying the latter in detail in (5.4) one obtains the solution in the (x, y) plane in explicit form.

Threshold-free orientational deformations are also possible. These arise when a magnetic field is applied to a pretwisted orientational nematic liquid crystal structure in the rectangle considered. The solution can be written most simply in terms of the angle α in the form of the ratio of elliptic Jacobi cosines

$$\mathrm{tg}\frac{\alpha}{2} = \pm\left[\frac{\nu_\eta^2 - \nu_\eta^4}{\nu_\mu^2 - \nu_\mu^4}\right]^{1/4}\mathrm{cn}\left[\frac{2\mu(x,y)}{\Delta\mu}K(\nu_\mu)\right]\Big/\,\mathrm{cn}\left[\frac{2\eta(x,y)}{\Delta\eta}K(\nu_\eta)\right] \tag{5.7}$$

Obviously, on two boundaries of the rectangular region ($\mu = \pm\mu_s$) the angle α vanishes, while on the two others ($\eta = \pm\eta_s$) it is equal to K. Consequently, there is a preliminary twist inside the region. The numbers ν_μ, ν_η obey the following characteristic equations

$$\frac{(1-2\nu_\eta^2)}{\Delta\eta^2}K^2(\nu_\eta)+\frac{(1-2\nu_\mu^2)}{\Delta\mu^2}K^2(\nu_\mu)=\frac{1}{4\mu_\chi^2},\quad \frac{\nu_\eta}{\Delta\eta^2}K^2(\nu_\eta)=\frac{\nu_\mu}{\Delta\mu^2}K^2(\nu_\mu) \tag{5.8}$$

In view of the fact that the quantities $K(\nu_\eta)$ and $K(\nu_\mu)$ are finite when $\nu_\eta^2 = \nu_\mu^2 = 1/2$, it follows from the first relation that $\Delta\eta = 0$, $\Delta\mu = 0$, i.e. deformations also exist when there are zero potential differences. Consequently, there are no bifurcations, but a lower boundary occurs in the spectrum of the numbers ν_η and ν_μ equal to $\sqrt{(1/2)}$.

Note that, in the limit, solution (5.4) describes a two-dimensional "kink", concentrated in the region of the boundaries of the closed volume, since sn → th as $\nu \to 1$. Solution (5.7) then reduces to a Jacobi delta-function [7], concentrated around the centre of the region. The corresponding localized orientational deformation, related to the initial twist, can be regarded as a specific defect of the orientational field. Since (5.4) and (5.7) were constructed from periodic functions, their extension outside the limits of the regions considered in the (η, μ) plane gives a regular "lattice" of localized defects. An orthogonal curvilinear grid is obtained in the (x, y) plane.

6. More general quasi-periodic solutions have recently been constructed in a number of publications, for example, in [10, 11], where the so-called method of finite-zone integration was employed. The solution is expressed in terms of generalized theta functions of two or more complex arguments $Z_n(x, y)$ ($n = 1, 2, \ldots$). The latter are governed by two or more fundamental periods, and when they are non-commensurable, they turn out to be quasi-periodic [10]. Fairly general solutions of the sine-Helmholtz and sine-Gordon equations are constructed on a many-sheeted Riemann surface and its parameters occur in the solution. Hence, despite the successes of the theory (the conditions for the solutions to be real, continuous and singular are established in [11]), some solutions have so far proved difficult to use for calculations. It is required, in particular, to calculate the Riemann matrix of the periods of a many-sheeted surface and to specify the latter in detail. Cases which arise when analysing two-zone ($n = 2$) solutions, when the theta functions depend on two periods, are relatively simple. As shown in [10, 11], in this case the Riemann surface can be arbitrary together with the matrix of the periods. The latter can be specified in general form as a symmetrical non-diagonalizable second-rank matrix with a positive real part.

The simplest quasi-periodic solution, which is expressed in terms of classical (single-zone) theta functions ϑ_1, ϑ_2, ϑ_3, is obtained in the form [10]

$$\alpha(\mu,\eta)=\frac{1}{i}\ln\frac{\vartheta_3(Z_1)\vartheta_3(Z_2)-\vartheta_2(Z_1)\vartheta_2(Z_2)}{\vartheta_3(Z_1)\vartheta_3(Z_2)+\vartheta_2(Z_1)\vartheta_2(Z_2)},\quad Z_k=\frac{z_k}{\kappa_k} \tag{6.1}$$

The arguments $z_1 = i\mu u_1 - \eta\upsilon_1$ and $z_2 = i\mu u_2 - \eta\upsilon_2$ include two periods υ_1 and υ_2. If these numbers are commensurable, the solution becomes a periodic function. The theta functions ϑ_1, ϑ_2, ϑ_3 can then be expressed in a well-known way [7] in terms of the Jacobi cosine and delta-function. Changing from (6.1) to $\mathrm{tg}(\alpha/2)$, we obtain one of the formulae of Section 5. The quasi-periodic solutions of more-general form are yet to be used in boundary-value problems.

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